Coloring is Harder than You Think

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COMPUTER SCIENCE
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Computer Science Theory

- Ask questions about what is even possible
- Sometimes like going back to kindergarten
  - Topics include: counting and coloring
- Today I’ll show you how to become a millionaire by coloring

Warm Up

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  - No beards! Only hairs on top of head!
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- Less than 200,000 hairs on any human head
- Roughly 6 million people in Maryland

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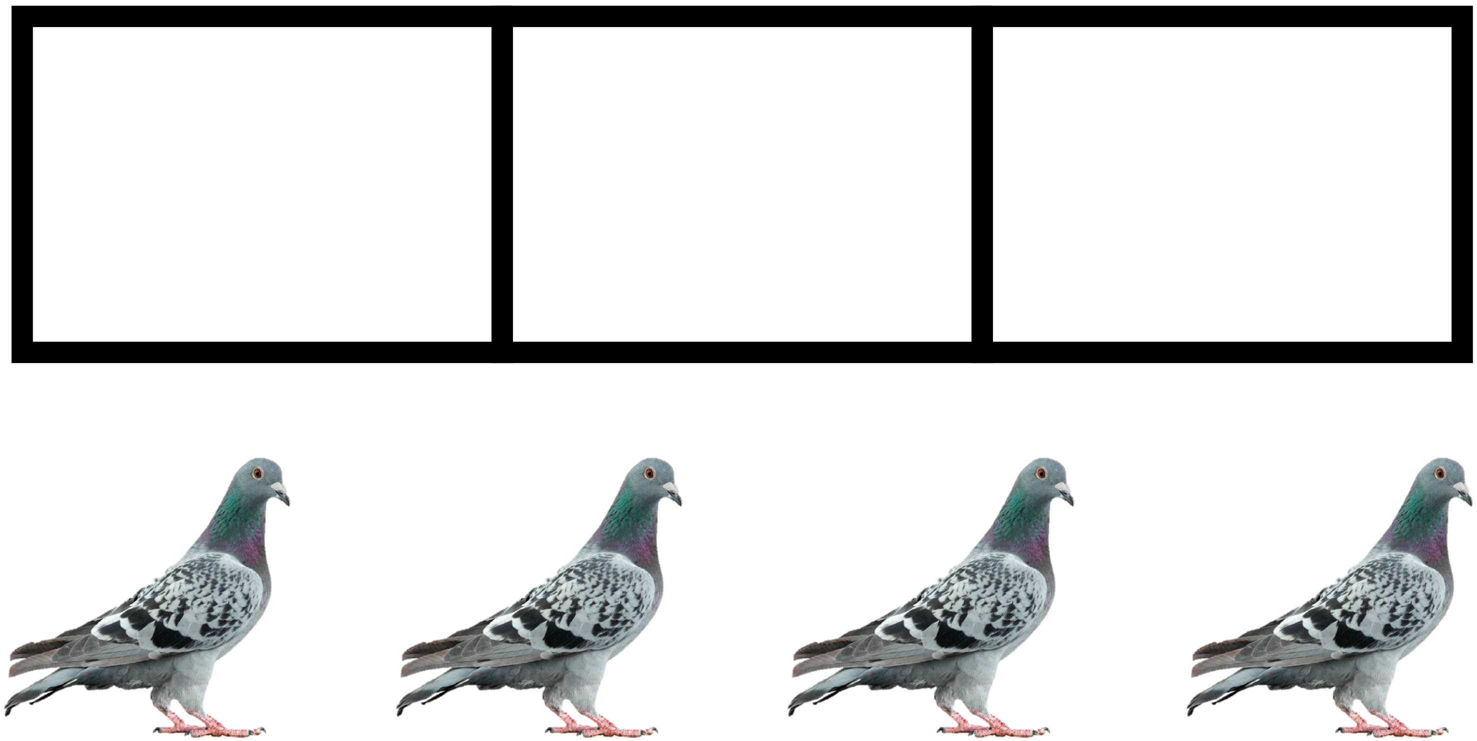

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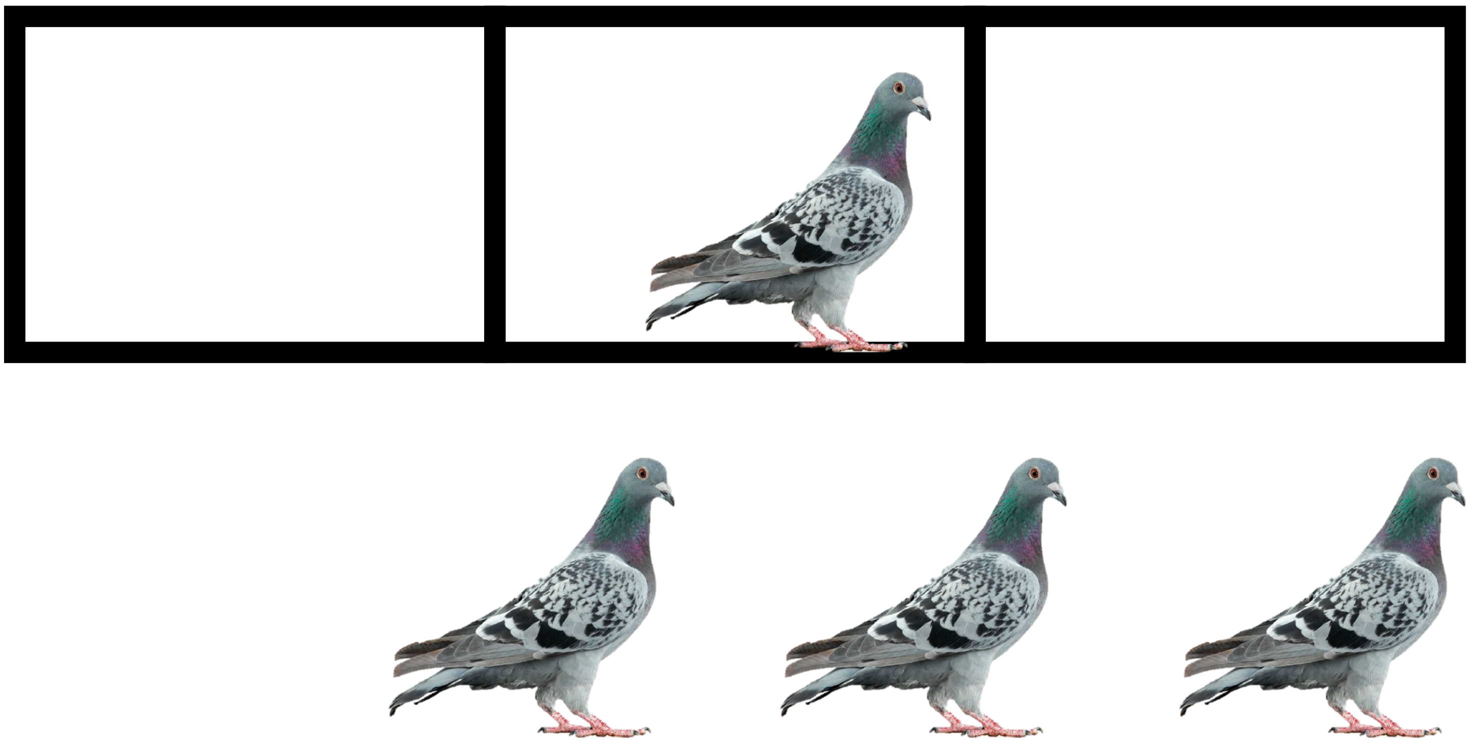

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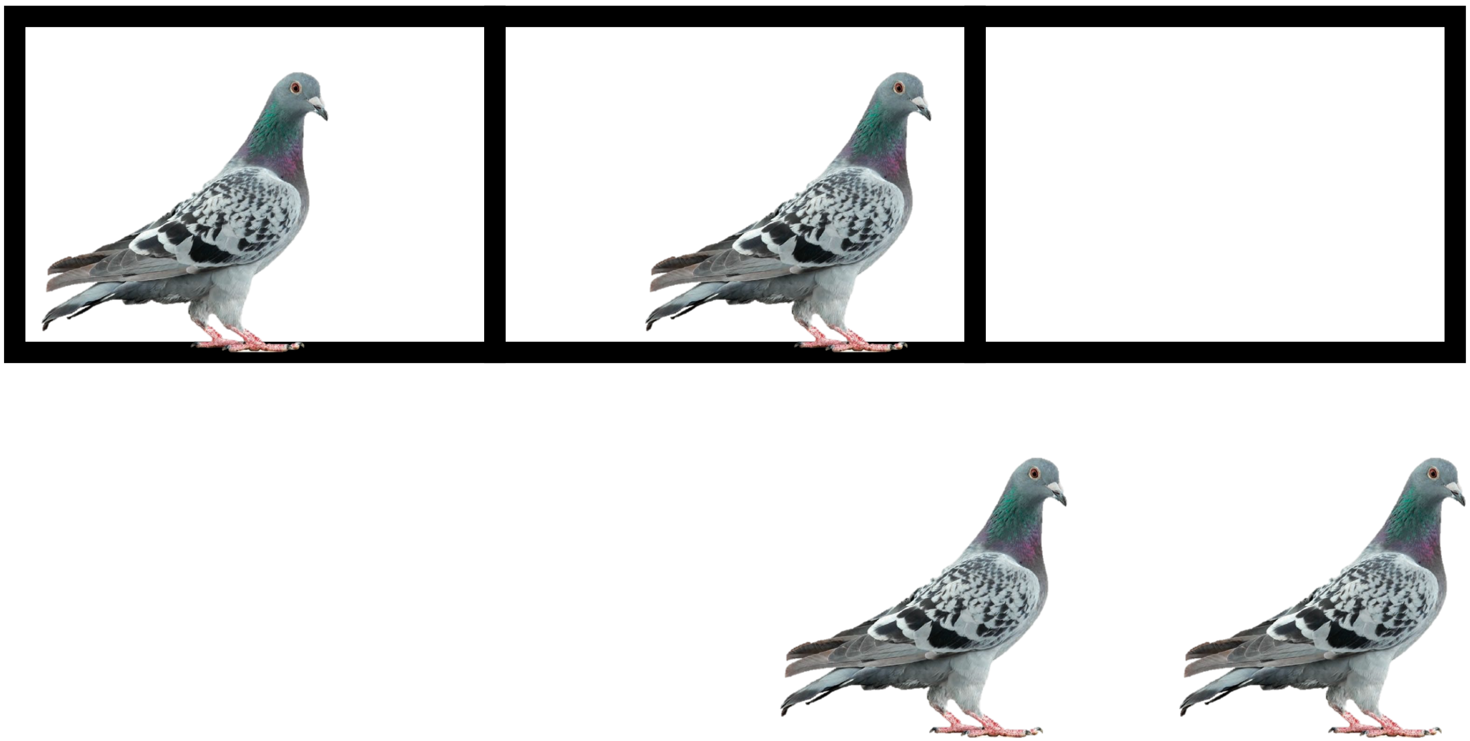

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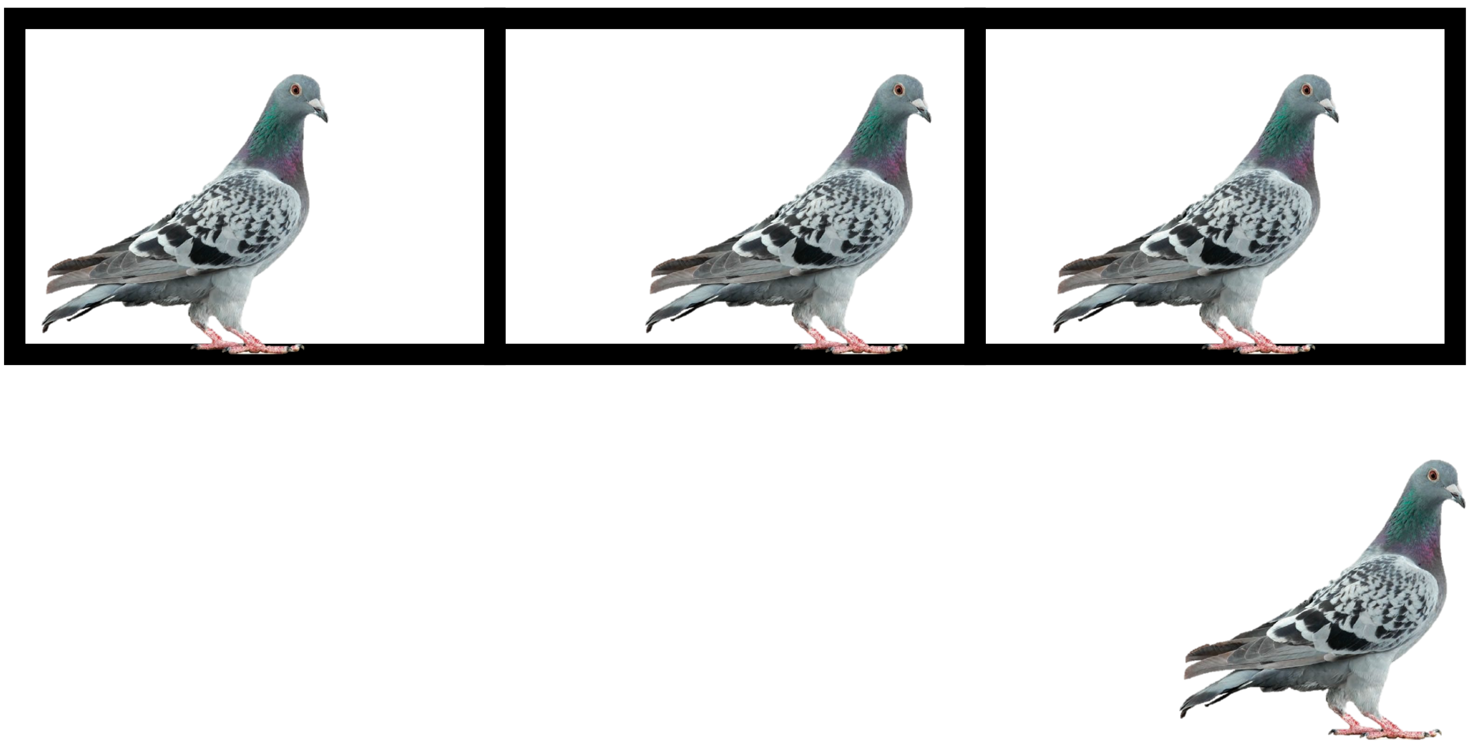

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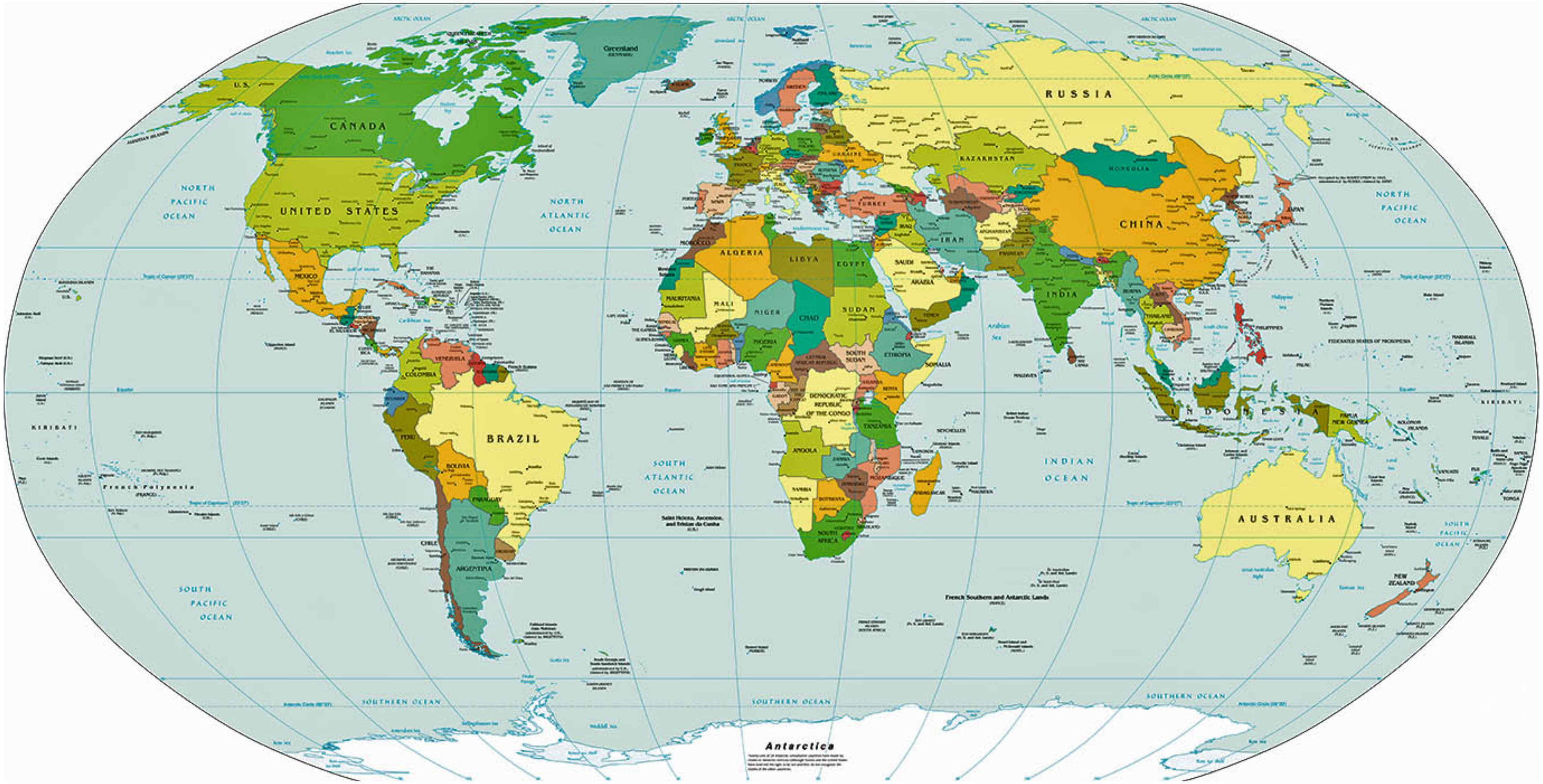
The Map Coloring Problem

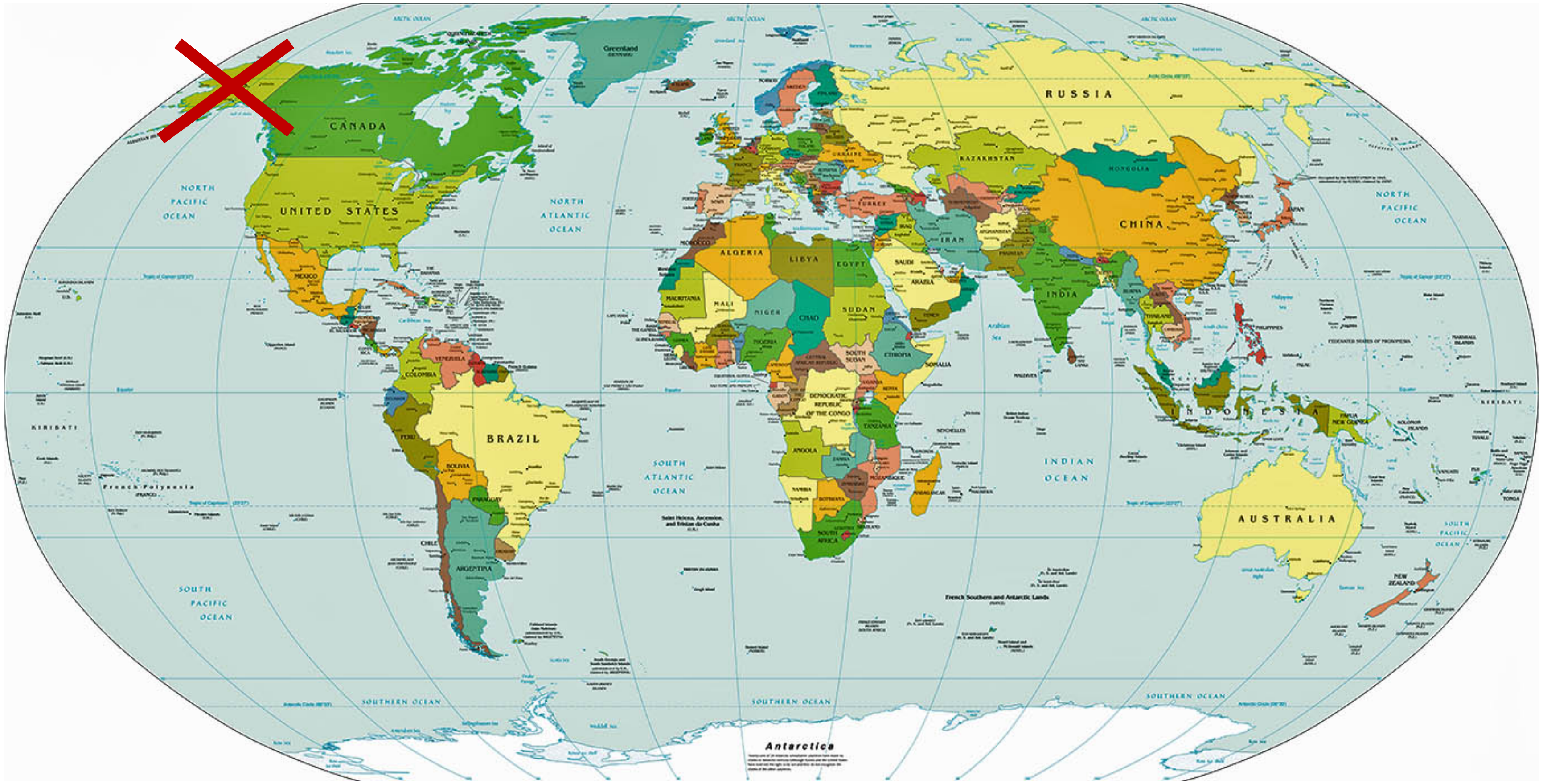
- What is the maximum number of colors needed to color any map so that no neighboring countries/states/counties are the same color?
  - i.e. how many different crayons do you need to buy to color any map?

One extra rule…



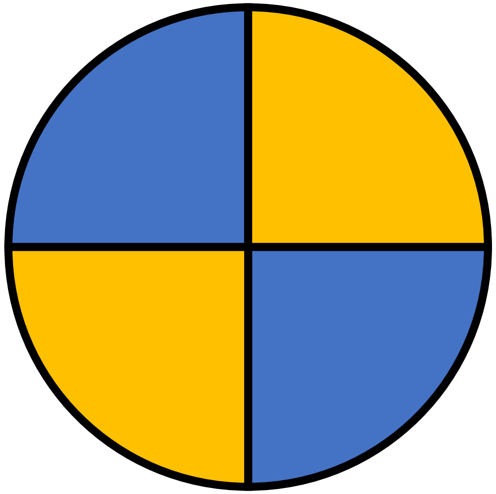
One extra rule… No Alaskas!

- Each country must be contiguous (one connected space)

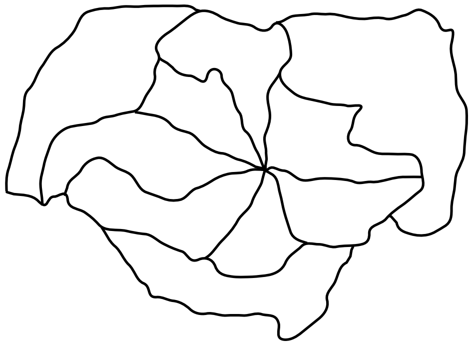


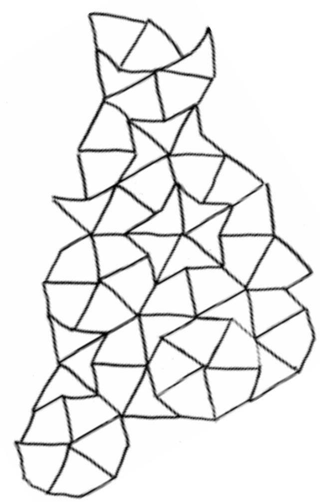
Color the maps with as few colors as possible

- Neighboring countries can't be the same color
- Corners touching doesn't count as neighboring
- How many colors do you need?



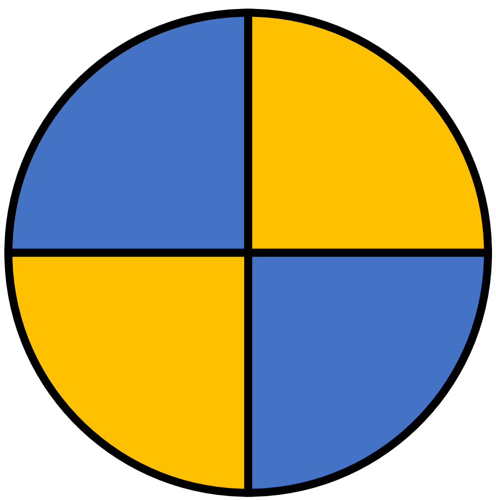
How many colors needed for each? Why?





Draw a map that needs as many colors as possible

- Neighboring countries can’t be the same color
- Corners touching doesn’t count as neighboring
- No Alaskas
- Don’t draw too many countries
- Can you draw a map that requires 3, 4, 5, 6, or more colors?



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  - Coloring the counties of England
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  - Asked: is four enough to color any map or do we need more?

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- 1976 Kenneth Appel and Wolfgang Haken
  - Any map (with no Alaskas) can be colored with four colors (no maps require five colors)
  - Checked 1,936 small maps using computer assistance
  - Proof is 400+ pages long (Checked by Dorothea Blostein)

4-coloring of US

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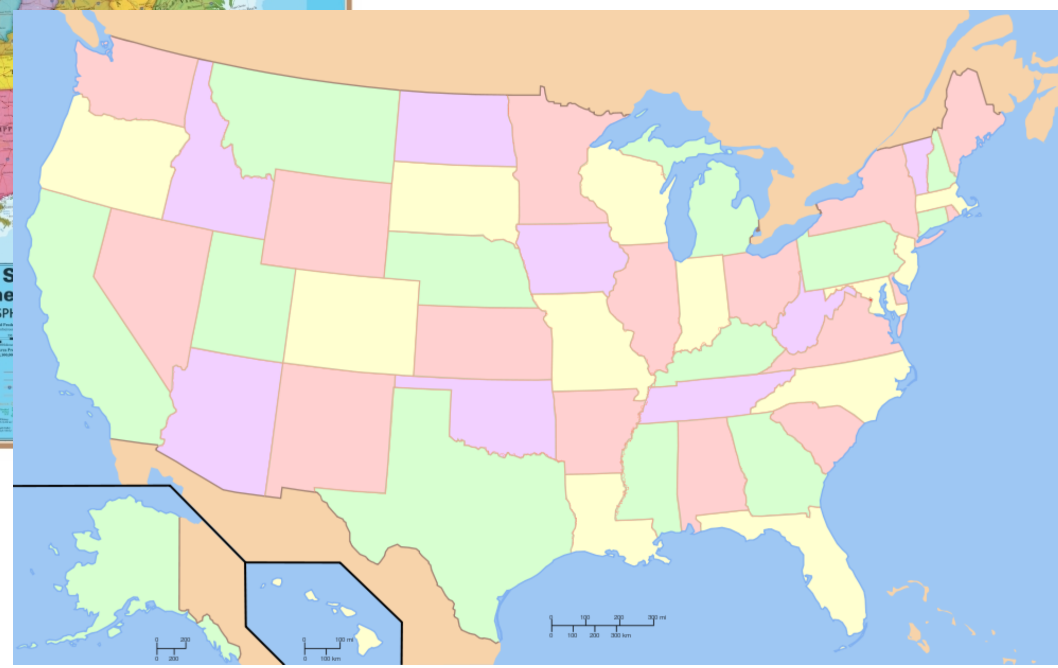
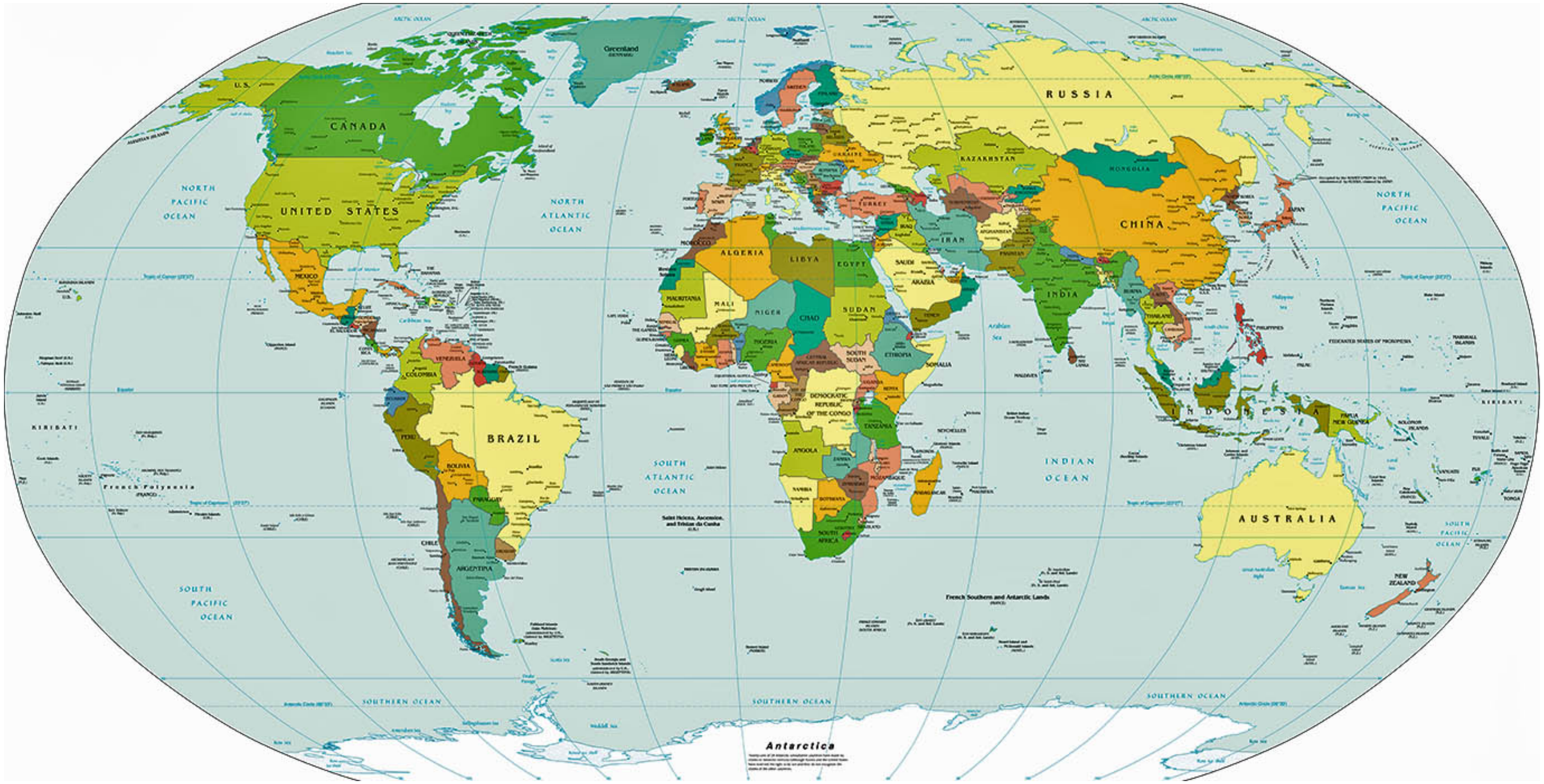


4-coloring of World



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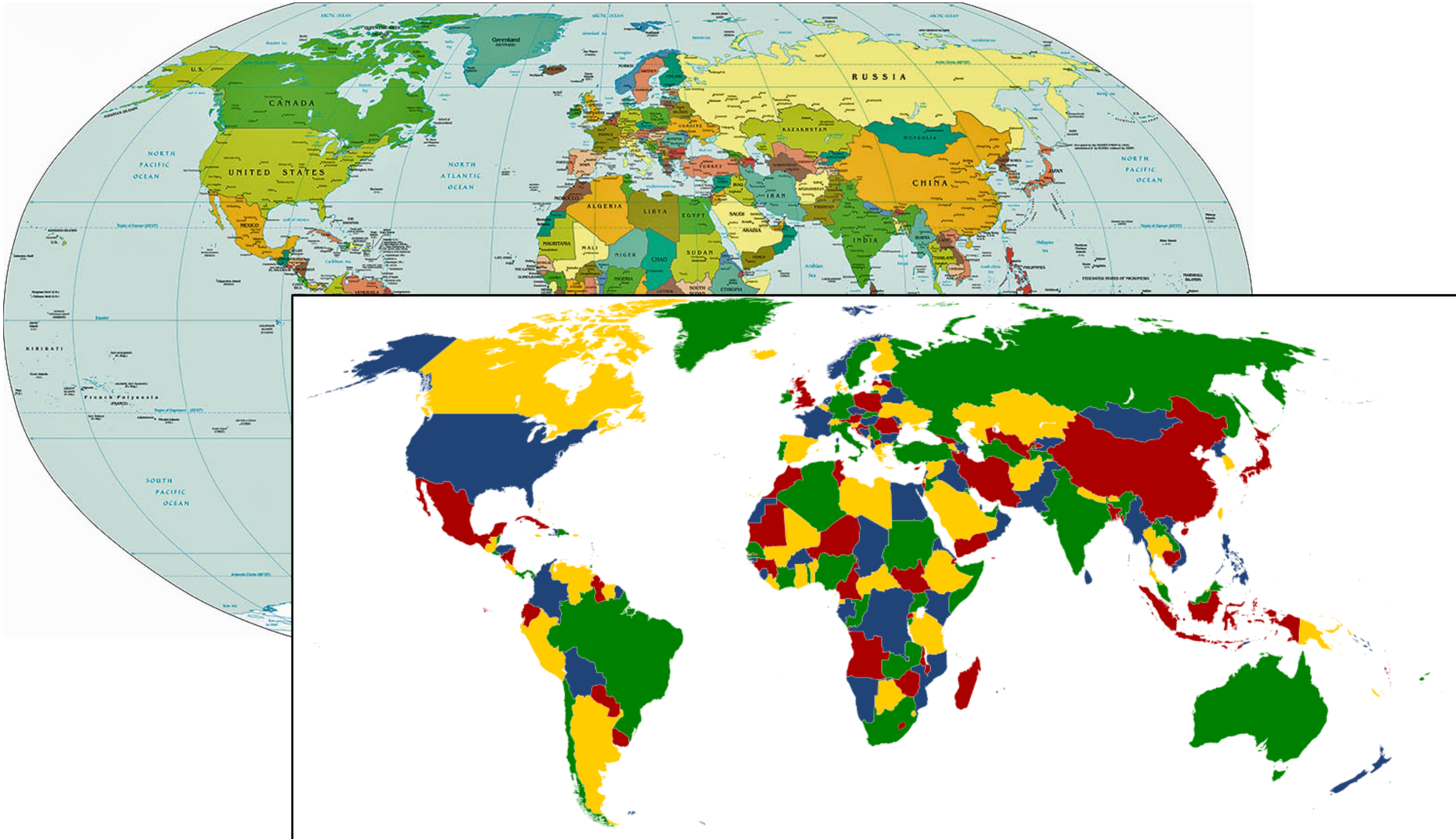
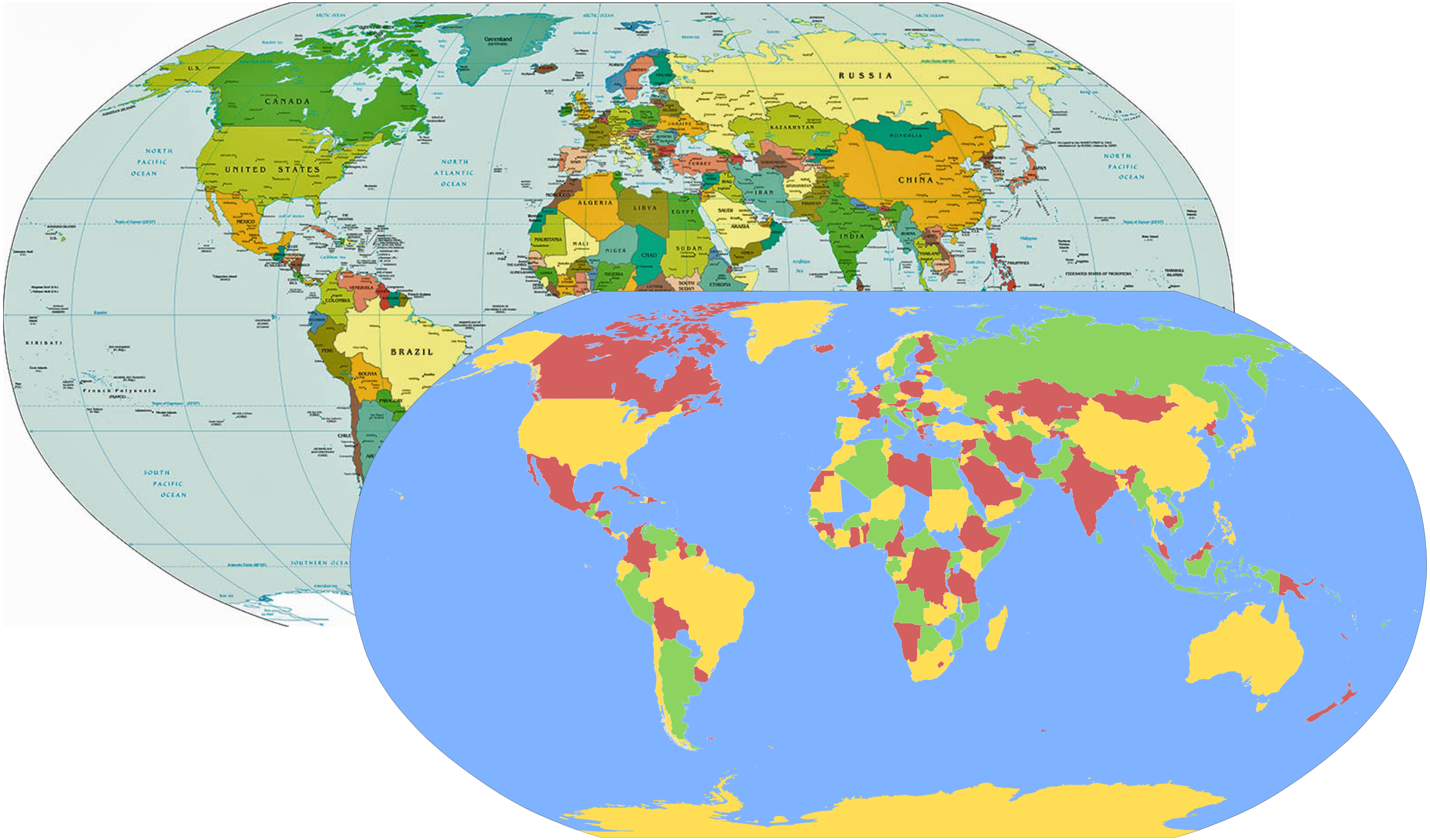


Including the ocean



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- The answer is unknown!

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- This would solve one of the seven Millennium Prize Problems which each carry a $1 million prize
  - Coloring is related to the P vs. NP problem
  - Only one of the other problems (Poincaré conjecture) has been solved, but they didn’t accept the prize money

P vs. NP (roughly)

- Is P = NP? → One of the biggest open problems in computer science
- P → the class of problems that a computer can solve efficiently
  - Example: Shortest path → find the shortest route between two cities
- NP → the class of problems where we can efficiently verify a solution
  - Example: Traveling Salesperson → Is there a route shorter than 1,000 miles that visits every city on a list?
  - If you give me a route, I can check it quickly
- By efficient, we mean that the running time is not exponential in the size of the problem
  - E.g. taking 3^n time to color n countries is not efficient
- The big question → If we can verify a solution to a problem quickly, does that imply we can also solve it quickly?

NP-complete

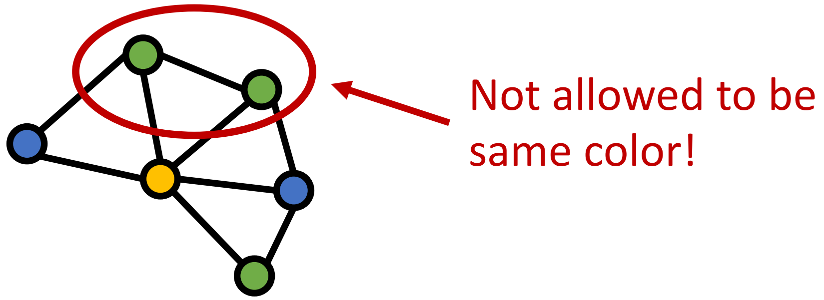
- NP-complete → Special class of problems
  - Known to be in NP
  - Not known if they are in P
  - Can all be transformed into each other → If you can solve one efficiently, you can solve any of them efficiently
  - Contains many fundamental important problems and even Sudoku
- All hard problems for computers to solve
  - In practice, we can sometimes solve these problems efficiently with "heuristics"
  - Heuristics are essentially algorithms that are not guaranteed to always work
  - Can also get approximate solutions

Is P = NP? What would this mean?

- Most computer scientists believe P is not equal to NP
  - Polls conducted by UMD professor Bill Gasarch!
  - However, we haven’t been able to prove this
- P = NP would imply many problems are much easier to solve than we think
- Downside of P = NP → our cryptography for secure internet transactions fails
- Upside of P = NP → many hard problems solvable from traveling salesperson to protein structure prediction
  - Theoretically, could then easily solve the other remaining Millennium Prize Problems

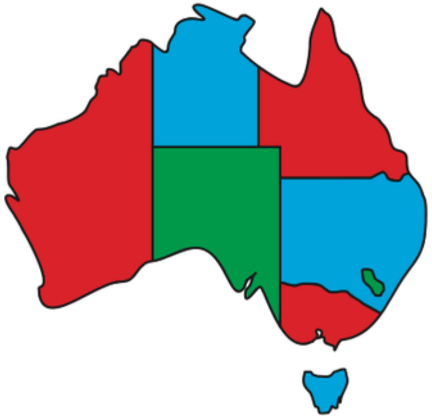
Graph Theory

- Graph → set of "vertices" (dots) with "edges" (lines) connecting them
- Graph coloring problem → Assign a color to each vertex such that vertices sharing an edge have different colors



Reductions: Transforming Problems into Each Other

- Notice that if we can solve graph coloring, then we can solve map coloring



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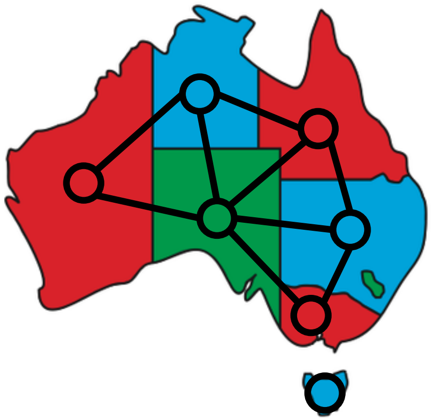

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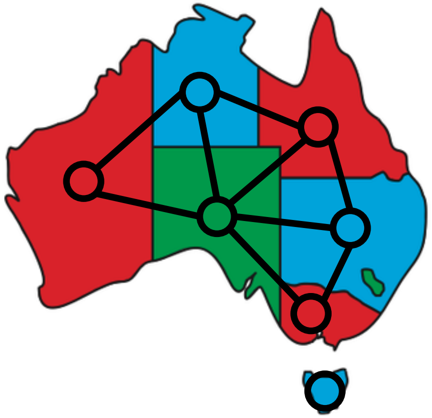
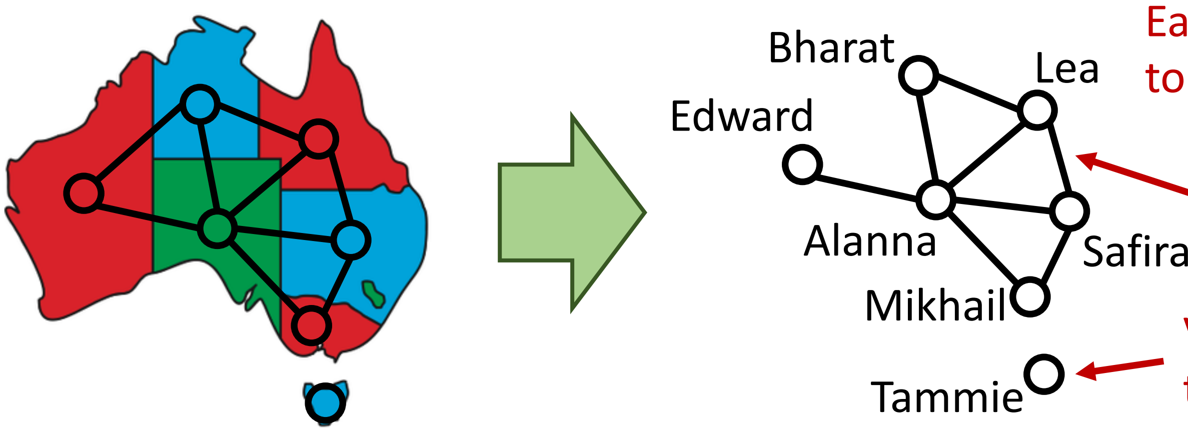


Reductions: Transforming Problems into Each Other

- Notice that if we can solve graph coloring, then we can solve map coloring
- What other problems can we solve with graph coloring?
  - Seating charts (your teach has a provably HARD job!)
  - Final exam scheduling on the fewest possible days
  - Any other NP-complete problem → Traveling salesperson, facility location, protein folding, etc.



Each "color" is a table to seat students at

Edges represent drama between two students

Vertices are students to seat at tables

Takeaways

- There are many problems that we think computers can't solve efficiently
- Can sometimes still solve them in practice using heuristics or approximations
  - Heuristic → Algorithm that is not guaranteed to work
  - Approximation → Algorithm that is guaranteed to work, but only gives an approximate solution
- When trying to improve a program, you should make sure that what you're doing is possible
- Many seemingly different problems can be transformed into each other through reductions
- What is P vs NP?
- How hard is coloring?

Thanks!